One-Page Summary of Integrals, by Joshua R. Davis, jdavis@math.wisc.edu, 31 Dec 2003

Integral calculus is motivated by the following question: Given a function $f(x)$, how large is the area between the graph $y = f(x)$ and the x-axis? To be precise, let $[a, b]$ be a closed interval, and assume that f is continuous and nonnegative (that is, never below the x-axis) on $[a, b]$; then what is the area of the region enclosed by $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$?

To answer this, we divide $[a, b]$ into n equal subintervals, and we inscribe the region with n rectangles, each with its base on one of the subintervals. If we add up the areas of these rectangles, then we get an approximate value for the area of the whole region under the graph. The more rectangles we use, the better our approximation should be. We define the *(definite) integral* $\int_a^b f(x)\, dx$ to be the limit of these approximations as the number n of rectangles goes to infinity; it is the exact area under the curve.
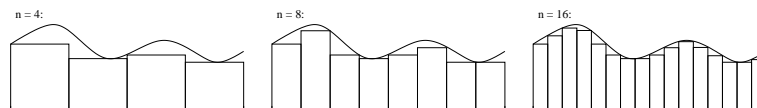
Areas are always positive, but integral calculus works better if we allow integrals to be negative. In particular, suppose that $y = f(x)$ does not stay above the x-axis, but encloses some areas below the axis. Then the integral counts these areas as negative. In other words, the integral is the area of the enclosed region above the axis, minus the area of the region below the axis. The true area enclosed by $y = f(x)$ and the axis is not $\int_a^b f(x)\, dx$ but $\int_a^b |f(x)|\, dx$.

By adding areas together, it is easy to see that $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ for any c in $[a, b]$. We can also integrate "backwards" over $[a, b]$: $\int_b^a f(x)\, dx$ is defined to be $-\int_a^b f(x)\, dx$.

The definition of definite integral is theoretically elegant but difficult to use in computations. Fortunately, calculus provides a powerful alternative, as follows. For any function f, the function F is said to be an *antiderivative* of f if f is the derivative of F. Of course, if $F' = f$, then for any constant C, $(F + C)' = F' + C' = f + 0 = f$, so we see that if F is an antiderivative, then so is $F + C$. It turns out that all antiderivatives of f are of the form $F + C$. We denote "the antiderivative" $F + C$ by $\int f(x)\, dx$.

The antiderivative of 0 is C. For $k \neq -1$, $\int x^k\, dx = x^{k+1}/(k + 1) + C$, while $\int x^{-1}\, dx = \ln|x| + C$. For $k > 0$, $\int k^x\, dx = k^x/(\ln k) + C$; in particular, $\int e^x\, dx = e^x + C$. Also, $\int \sin x\, dx = -\cos x + C$, and $\int \cos x\, dx = \sin x + C$. In general, antiderivatives can be very difficult to compute, but these rules help:

Theorem. Let k be a constant, and let f and g be two continuous functions. Then
  A. $\int f + g\, dx = \int f\, dx + \int g\, dx$,
  B. $\int k \cdot g\, dx = k \cdot \int g\, dx$,
  C. $\int f \cdot g' + g \cdot f'\, dx = f \cdot g + C$ (this is called *integration by parts*), and
  D. $\int f'(g) \cdot g'\, dx = f(g) + C$ (this is called *integration by substitution*).

For example, say $f(x) = \sin x$ and $g(x) = 3x^2$, so that $f(g) = \sin(3x^2)$ and $(f(g))' = \cos(3x^2) \cdot 6x$, by the chain rule. Integration by substitution "undoes" the chain rule: $\int \cos(3x^2) \cdot 6x\, dx = \sin(3x^2) + C$.

Similarly, integration by parts undoes the product rule of differentiation. The formula in part C is often rewritten as $\int f \cdot g'\, dx = f \cdot g - \int g \cdot f'\, dx$. For example, it tells us that $\int x \cos x\, dx = x \cdot \sin x - \int 1 \cdot \sin x\, dx$; since $\int 1 \cdot \sin x\, dx = \int \sin x\, dx = -\cos x + C$, we see that $\int x \cos x\, dx = x \sin x + \cos x + C$.

The following crucial theorem connects derivatives, antiderivatives, and definite integrals.

The Fundamental Theorem of Calculus. Let f be a continuous function on $[a, b]$. Then
  A. $\int_a^b f(x)\, dx = F(b) - F(a)$ for any antiderivative F of f, and
  B. $\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$ for any x in $[a, b]$. That is, $\int_a^x f(t)\, dt$ is an antiderivative of $f(x)$.

Part A gives us a way to compute definite integrals, and earns the antiderivative its alternative name, the *indefinite integral*. For example, suppose we want to compute $\int_1^2 x^2\, dx$. An antiderivative of x^2 is $x^3/3$, so part A tells us that $\int_1^2 x^2\, dx = 2^3/3 - 1^3/3 = 7/3$.