One-Page Summary of Limits and Continuity, by Joshua R. Davis, jdavis@math.wisc.edu, 31 Dec 2003

The concept of *limit* is fundamental in calculus; for example, the definitions of *derivative* and *integral* depend on it. A limit is a way of describing the behavior of a function f as its input x gets very close to a fixed number c. The limit $\lim_{x \to c} f(x)$ has value L if, no matter how close we want the graph $y = f(x)$ to stay to the horizontal line $y = L$, we can always restrict the domain of f to a small enough region of x-values around c to make the graph stay as close to the line as we wanted. In other words, we want $f(x)$ to stay within a distance of ϵ of L — that is, we want $|f(x) - L| < \epsilon$ — and we are allowed to restrict ourselves to x-values within δ of c — that is, $|x - c| < \delta$ — to make it. For the limit to be L, such a number δ must exist for every value of ϵ, no matter how small.

Definition. $\lim_{x \to c} f(x) = L$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all values of x (in the domain of f) satisfying $0 < |x - c| < \delta$.

Note that the definition requires $0 < |x - c|$, which implies that x does not equal c. The limit is only concerned with the behavior of the function *near* c, not *at* c. If the behaviors near c and at c agree, then the function is said to be continuous there.

Definition. A function f is *continuous at* c if $\lim_{x \to c} f(x) = f(c)$. (In other words, f is continuous at c if we can compute its limit at c by just plugging c into f. Not all functions are so nice!) We say that f is *continuous* if it is continuous at every point of its domain.

Intuitively, a function is continuous if its graph is unbroken, steady, predictable, etc. For example, it is not hard to show that for any constant k, $\lim_{x \to c} k = k$; it is also true that $\lim_{x \to c} x = c$. Thus the constant function $f(x) = k$ and the identity function $f(x) = x$ are continuous.

Of course, a function fails to be continuous where it is undefined and where its limit does not exist. For example, define $g(x)$ to be 0 for $x \leq 0$ and 1 for $x > 0$. For this function it is useful to analyze the limit at 0 from the left and right sides independently. In general the *right-hand limit* $\lim_{x \to c^+} f(x) = L$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x satisfying $0 < x - c < \delta$. The *left-hand limit* $\lim_{x \to c^-} f(x)$ is defined similarly, with $-(x - c)$ substituted for $x - c$. For the full limit $\lim_{x \to c} f(x)$ to exist, the left- and right-hand limits must exist and must agree. Returning to our example, $\lim_{x \to 0^-} g(x) = 0$ and $\lim_{x \to 0^+} g(x) = 1$, but the full limit $\lim_{x \to 0} g(x)$ does not exist, since the one-sided limits don't agree. Since its limit does not exist at 0, g cannot be continuous there, even though it is defined.

Theorem. Limits respect arithmetic: If $\lim_{x \to c} f_1(x) = L_1$ and $\lim_{x \to c} f_2(x) = L_2$, then

A. $\lim_{x \to c}(f_1(x) + f_2(x)) = L_1 + L_2$,

B. $\lim_{x \to c} k \cdot f_1(x) = k \cdot L_1$ for any constant k,

C. $\lim_{x \to c} f_1(x) \cdot f_2(x) = L_1 \cdot L_2$, and

D. $\lim_{x \to c} f_1(x)/f_2(x) = L_1/L_2$, provided $L_2 \neq 0$.

Since $\lim_{x \to c} x$ is c, part C of the theorem tells us that the limit (as $x \to c$) of $x \cdot x = x^2$ is c^2, and part B tells us that the limit of $3 \cdot x^2$ is $3c^2$. Then from part A we know that the limit of $3x^2 + x$ is $3c^2 + c$; that is, $3x^2 + x$ is continuous. By similar reasoning it is easy to prove that every polynomial is continuous.

A *rational function* is one that can be written as a quotient of two polynomials, such as $(3x^2+x)/(2x-1)$; since polynomials are continuous, part D tells us that any rational function is continuous wherever its denominator is nonzero. It is also true (but harder to prove) that the trigonometric functions $\sin x$ and $\cos x$ are continuous; part D then tells us that $\tan x = \sin x / \cos x$ is continuous wherever $\cos x$ is nonzero.

Another theorem says that the composition of any two continuous functions is continuous. For example, since $\sin x$ and $3x^2 + x$ are continuous, so are the compositions $\sin(3x^2 + x)$ and $3(\sin x)^2 + \sin x$.

So far we have discussed limits only at a *number* c. We said that x was "close to c" when $|x - c|$ was less than some small number δ. Now, to define limits at ∞, we make x "close to ∞" by making it greater than some large number N. For example, $\lim_{x \to \infty} f(x) = L$ means that, for every $\epsilon > 0$, there exists an $N > 0$ such that $|f(x) - L| < \epsilon$ for all $x > N$. Similarly, $\lim_{x \to \infty} f(x) = \infty$ means that, for every $M > 0$, there exists an $N > 0$ such that $f(x) > M$ for all $x > N$. Limits at $-\infty$ are analogous.