# CS 202 Fall 2014
# Exam #1, 10 October

Name: ______________________________________

You have 60 minutes for this exam.

Read briefly through the whole exam before you start.

Don't spend too much time on any one question.

No notes, calculator, etc. are allowed.

Except where otherwise indicated, always show your work or otherwise explain your answer. A correct answer with no supporting argument may not earn much credit.

Carefully indicate your answers — draw boxes around them to distinguish them from your other work. Please write legibly.

If the instructions say to circle something, actually circle it; don't rewrite it yourself.

If you think that a question is unclear or ambiguous (or if you think that there is an error), make a good-faith effort to interpret the intent of the question, and explain your interpretation in your solution.

Good luck!

| Question: | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Points: | 10 | 8 | 12 | 12 | 42 |
| Score: | | | | | |

*(10 pts.)* 1. For this question, our domain of discourse is all people who have ever lived. We define two predicates:

- isMother$(x, y)$ is true if and only if $x$ is the mother of $y$.
- isTaller$(x, y)$ is true if and only if $x$ is *at least as tall* as $y$. (So it's true for ties too.)

For the purposes of our question, we assume that somebody in history was the strictly shortest ever, and somebody else was the strictly tallest ever. We also assume that each person has exactly one mother.

Below are several sentences of predicate logic, with a predicate $P$ in them. For each one, circle the predicates that can replace $P$ and make the sentence true about the "real world" (subject to our assumptions): either isMother or isTaller, or both, or neither. If neither, circle that. You do not have to explain your answer.

(a) $\forall a : \exists b : P(b, a)$.

**isMother** **isTaller** **Neither**

(b) $\forall a : \exists b : P(a, b)$.

**isMother** **isTaller** **Neither**

(c) $\forall c : \exists a : \exists b : P(a, b) \wedge P(b, c) \wedge \neg P(a, c)$.

**isMother** **isTaller** **Neither**

(d) $\exists a : \forall b : P(b, a)$.

**isMother** **isTaller** **Neither**

(e) $\exists c : \exists a : \exists b : (a \neq b) \wedge P(c, a) \wedge P(c, b)$.

**isMother** **isTaller** **Neither**

2. Simplify each proposition down to as few symbols as possible. You may use variables, $\neg$, $\wedge$, $\vee$, $\oplus$, $\Rightarrow$, $\Leftrightarrow$, True, False, parentheses, and square brackets. Parentheses and square brackets have the same meaning and don't count toward your symbol count. All the others count for one symbol each.

*(4 pts.)* (a) $\neg[(\neg p \wedge \neg q) \vee (p \vee q)]$

*(4 pts.)* (b) $\neg(p \Rightarrow r) \vee (p \wedge [(q \wedge r) \vee \neg(r \vee \neg q)])$

*(6 pts.)* 3. (a) Prove that $n^3 + 2n$ is divisible by 3 for all positive integers $n$.

*(6 pts.)* (b) Prove that the sum of the squares of any two consecutive positive integers is odd.

4. In each part below, there is a statement, followed by a claim about that statement. After that is a proof of the claim. For each one, circle **Valid** or **Invalid** to indicate whether the proof is valid for the claim. If you circle "Invalid", concisely justify your answer.

*Note: there are no arithmetic or algebra errors in these proofs. You don't have to double-check the expansion of* $(\ldots)^4$*, or the prime factorizations, or the factorization of the polynomial; you can take my word for it that they are correct.*

*(4 pts.)* (a) *Statement:* For every positive integer $n$, $n^4 - n$ is even. *Claim:* This statement is true.

*Proof:* Since $n$ is an integer $> 0$, we can write it as $2k \pm 1$ for some integer $k \geq 0$. We proceed by cases:

- $n = 2k - 1$. Then $n^4 - n = (2k-1)^4 - (2k-1) = 16k^4 - 32k^3 + 24k^2 - 8k + 1 - 2k + 1 = 2 \cdot (8k^4 - 16k^3 + 12k^2 - 5k + 1)$. Since the expression inside the parentheses is a sum of products of integers, it is itself an integer. 2 times an integer is even, and so $n^4 - n$ is even.
- $n = 2k + 1$. Then $n^4 - n = (2k+1)^4 - (2k+1) = 16k^4 + 32k^3 + 24k^2 + 8k + 1 - 2k - 1 = 2 \cdot (8k^4 + 16k^3 + 12k^2 + 3k)$. Since the expression inside the parentheses is a sum of products of integers, it is itself an integer. 2 times an integer is even, and so $n^4 - n$ is even.

Is this proof **Valid** or **Invalid**? *(Circle one; if invalid, explain why.)*

*(4 pts.)* (b) *Statement:* Every odd positive integer is within 2 of a whole power of some prime number.

*Claim:* This statement is false. *Proof:* 93 is 4 greater than the next lower power of a prime ($89 = 89^1$) and 4 less than the next higher power of a prime ($97 = 97^1$). 93 ($= 3 \cdot 31$) is not itself a power of a prime, nor are 91 ($= 7 \cdot 13$), 92 ($= 2^2 \cdot 23$), 94 ($= 2 \cdot 47$) or 95 ($= 5 \cdot 19$).

Is this proof **Valid** or **Invalid**? *(Circle one; if invalid, explain why.)*

*(4 pts.)* (c) *Statement:* $\forall x > 6,\ x^2 > 10x - 24$. *Claim:* This statement is true.

*Proof:* If $x^2 > 10x - 24$, then $x^2 - 10x + 24 > 0$, which means that $(x-6)(x-4) > 0$; call this final inequality the claim $B$. $x > 6$ makes both factors $> 0$. The product of two positive numbers is positive, so $B$ is true, which means the original statement is true.

Is this proof **Valid** or **Invalid**? *(Circle one; if invalid, explain why.)*