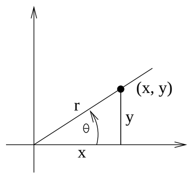
One-Page Summary of Trigonometry, by Joshua R. Davis, jdavis@math.wisc.edu, 31 Dec 2003

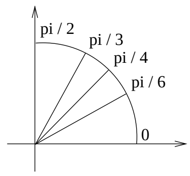
Angles describe rotation. Rotation can be measured in terms of revolutions or degrees, but mathematics works best when we measure angles in radians. One revolution equals $360°$, which equals 2π radians. (Recall that 2π is the circumference of a unit circle; it is approximately 6.28.) For example, $1/4$ rev $= 90° = \pi/2$. By convention, positive angles describe counterclockwise rotation, and negative ones describe clockwise rotation.

When drawing an angle θ, we typically show where it rotates the positive x-axis to. This rotated ray is called the *terminal ray* of the angle. Now let (x, y) be any point on the terminal ray, and let $r = \sqrt{x^2 + y^2}$ be the distance from the origin to this point. The *cosine* and *sine* of θ are

$$\cos(\theta) = x/r \quad \text{and} \quad \sin(\theta) = y/r.$$

They depend only on θ, not on the particular choice of (x, y) on the terminal ray. For any θ, cosine and sine are always between -1 and 1. There are four other, less important, trigonometric functions, defined in terms of cosine and sine: $\tan(\theta) = \sin(\theta)/\cos(\theta)$, $\sec(\theta) = 1/\cos(\theta)$, $\csc(\theta) = 1/\sin(\theta)$, and $\cot(\theta) = 1/\tan(\theta)$.





You will want to memorize the cosine and sine of a few angles between 0 and $\pi/2$: $\cos(0) = 1$, and $\sin(0) = 0$; $\cos(\pi/6) = \sqrt{3}/2$, and $\sin(\pi/6) = 1/2$; $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$; $\cos(\pi/3) = 1/2$, and $\sin(\pi/3) = \sqrt{3}/2$; $\cos(\pi/2) = 0$, and $\sin(\pi/2) = 1$. Then you can find the trig values at many other angles using these basic identities, which follow from the definitions and the symmetries of a circle:

- $\cos(\theta) = \sin(\pi/2 - \theta)$.
- $\cos(-\theta) = \cos(\theta)$, and $\sin(-\theta) = -\sin(\theta)$.
- $\cos(\pi - \theta) = -\cos(\theta)$, and $\sin(\pi - \theta) = \sin(\theta)$.
- For any integer k, $\cos(\theta + k2\pi) = \cos(\theta)$ and $\sin(\theta + k2\pi) = \sin(\theta)$.

Given a number t such that $-1 \leq t \leq 1$, we might wonder which angles θ have cosine equal to t. If $t = 1$, then clearly $\theta = 0$ works; if $t = -1$, then $\theta = \pi$ works. Otherwise, there are two basic values of θ such that $\cos(\theta) = t$; one is between 0 and π, and the other is between π and 2π. We define the *inverse cosine* $\cos^{-1}(t)$ to be the unique angle θ such that $\cos(\theta) = t$ and $0 \leq \theta \leq \pi$. So the two basic solutions of $\cos(\theta) = t$ are $\theta = \cos^{-1}(t)$ and $\theta = 2\pi - \cos^{-1}(t)$. We can add 2π to any solution to get another solution, so really there are two "chains" of solutions: $\theta = \cos^{-1}(t) + k2\pi$ and $\theta = 2\pi - \cos^{-1}(t) + k2\pi$.

Similarly, the *inverse sine* $\sin^{-1}(t)$ is the unique angle θ such that $\sin(\theta) = t$ and $-\pi/2 \leq \theta \leq \pi/2$. The two chains of solutions of $\sin(\theta) = t$ are $\theta = \sin^{-1}(t) + k2\pi$ and $\theta = \pi - \sin^{-1}(t) + k2\pi$. For example, suppose that $\sin(3\theta) = 0.2$. We deduce that $3\theta = \sin^{-1}(0.2) + k2\pi$ or $3\theta = \pi - \sin^{-1}(0.2) + k2\pi$. Thus $\theta = \sin^{-1}(0.2)/3 + k2\pi/3$ or $\theta = \pi/3 - \sin^{-1}(0.2)/3 + k2\pi/3$. There are six solutions between 0 and 2π.

By convention, we write $\cos^2(\theta)$ as shorthand for $(\cos(\theta))^2$. It does *not* mean $\cos\cos(x)$. For any $n \geq 0$, $\cos^n(\theta)$ is shorthand for $(\cos(\theta))^n$. But note well that $\cos^{-1}(t) \neq (\cos(t))^{-1}$! Similar remarks apply for sine. The Pythagorean theorem now yields

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

In general, statements like $\cos(\theta + \phi) = \cos(\theta) + \cos(\phi)$ are *not* true. Instead, we have sum formulae:

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi);$$
$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi).$$

Formulae for $\cos(\theta - \phi)$ and $\cos(2\theta)$ follow by replacing ϕ with $-\phi$ and θ, respectively.

Consider a triangle with sides of length a, b, c and angles of size α, β, γ (opposite a, b, c, respectively). The *law of sines* and the *law of cosines* are

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \qquad \text{and} \qquad c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

When $\gamma = \pi/2$, these reduce down to the definition of sine and the Pythagorean theorem, $c^2 = a^2 + b^2$.