One-Page Summary of Vectors, by Joshua R. Davis, jdavis@math.wisc.edu, 31 Dec 2003

A *vector* is a quantity that has both magnitude and direction. Vectors are used everywhere in physics to describe displacement (such as "37 meters northwest"), force ("12 pounds down"), etc. We will use symbols like $\vec{v}$ and $\vec{w}$ represent vectors, and symbols like c represent ordinary numbers, which we call *scalars*.

Let's begin by describing two-dimensional vectors using the physical notion of displacement. Each vector $\vec{v}$ is a pair $\langle v_1, v_2 \rangle$ of numbers. This $\vec{v}$ represents a displacement of v_1 units in the x-direction and v_2 units in the y-direction. The *length* (or *magnitude*) of $\vec{v}$ is denoted $|\vec{v}|$. By the Pythagorean theorem, $|\vec{v}| = \sqrt{v_1^2 + v_2^2}$. For example, $\langle 3, 4 \rangle$ means "move 3 units to the right and 4 up". The total displacement is $\sqrt{3^2 + 4^2} = 5$, in a vaguely up-right direction. Similarly, $\langle -2, \pi \rangle$ represents a displacement of "2 left and π up".

There are two basic operations on vectors: addition and scalar multiplication. To add two vectors $\vec{v}$ and $\vec{w}$, you simply add their components independently:

$$\langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle = \langle v_1 + w_1, v_2 + w_2 \rangle.$$

Obviously, adding $\langle 0, 0 \rangle$ to a vector doesn't change it; $\langle 0, 0 \rangle$ is the *zero vector*, $\vec{0}$. Now, to muliply a vector $\vec{v}$ by a scalar c, you multiply c across the components of $\vec{v}$:

$$c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle.$$

For example, $1\vec{v} = \vec{v}$ and $0\vec{v} = \vec{0}$. The effect of scalar multiplication is to stretch $\vec{v}$ by a factor of c. Vectors of length 1 are called *unit vectors*. If $\vec{v} \neq \vec{0}$, then $\vec{v}/|\vec{v}|$ is the unit vector pointing in the direction of $\vec{v}$. (When we write $\vec{v}/c$, we mean $(1/c)\vec{v}$.) The negation $-\vec{v} = -1\vec{v}$ is the vector with the same length as $\vec{v}$, but pointing in the opposite direction. Notice that $\vec{v} + -\vec{v} = \vec{0}$; we define vector subtraction by $\vec{v} - \vec{w} = \vec{v} + -\vec{w}$.

One might be tempted to define vector multiplication as $\langle v_1, v_2 \rangle \langle w_1, w_2 \rangle = \langle v_1 w_1, v_2 w_2 \rangle$, but it turns out that this operation is neither useful nor interesting. What is useful is the *dot product*,

$$\langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle = v_1 w_1 + v_2 w_2.$$

Notice that the result is a scalar, not a vector! The dot product is important for several related reasons:

- The length of $\vec{v}$ equals $\sqrt{\vec{v} \cdot \vec{v}}$. In other words, $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$.
- If θ is the angle between $\vec{v}$ and $\vec{w}$, then $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta)$. If $|\vec{v}| = |\vec{w}| = 1$, then $\theta = \cos^{-1}(\vec{v} \cdot \vec{w})$.
- $\vec{v} \cdot \vec{w} = 0$ if and only if $\vec{v}$ and $\vec{w}$ are perpendicular (or one of them is zero).

All of this generalizes to higher dimensions. For any positive integer n, we define an n-dimensional vector $\vec{v}$ to be an ordered n-tuple of numbers, $\langle v_1, v_2, v_3, \ldots, v_n \rangle$. The operations are

$$\begin{aligned} \vec{v} + \vec{w} &= \langle v_1, \ldots, v_n \rangle + \langle w_1, \ldots, w_n \rangle = \langle v_1 + w_1, \ldots, v_n + w_n \rangle, \\ c\vec{v} &= c\langle v_1, \ldots, v_n \rangle = \langle cv_1, \ldots, cv_n \rangle, \\ \vec{v} \cdot \vec{w} &= \langle v_1, \ldots, v_n \rangle \cdot \langle w_1, \ldots, w_n \rangle = v_1 w_1 + \ldots + v_n w_n. \end{aligned}$$

The zero vector is $\langle 0, 0, 0, \ldots, 0 \rangle$. Vector addition is associative and commutative. Also, vectors enjoy

- Scalar Distributivity: $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$.
- Scalar Associativity: $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$.
- Commutativity: $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
- Distributivity: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

Although n-dimensional space is hard to visualize for $n > 3$, vectors make it easy to talk about. For example, the length of the vector $\vec{v}$ is $\sqrt{\vec{v} \cdot \vec{v}}$, and the angle between two unit vectors $\vec{v}$ and $\vec{w}$ is $\cos^{-1}(\vec{v} \cdot \vec{w})$.

A peculiar feature of three-dimensional vectors is the *cross product*, defined as

$$\langle v_1, v_2, v_3 \rangle \times \langle w_1, w_2, w_3 \rangle = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle.$$

Notice that the result is a vector, not a scalar. Here are some of its properties.

- Scalar Associativity: $(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w})$.
- Anticommutativity: $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.
- Distributivity: $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
- $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.
- If θ is the angle between $\vec{v}$ and $\vec{w}$, then $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\theta)$.
- $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$.
- $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{w} \times \vec{v}) \times \vec{u} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.